NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2501

EXPRESSIONS FOR MEASURING THE ACCURACY OF APPROXIMATE SOLUTIONS TO COMPRESSIBLE FLOW THROUGH CASCADES OF BLADES WITH EXAMPLES OF USE

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SUMMARY

Four necessary conditions for steady irrotational compressible flow through a straight cascade of blades are derived from the irrotationality condition and the conservation of mass and momentum. Expressions are obtained which measure the degree to which an approximate compressible-flow solution departs from these conditions. The expressions may be used as a measure not only of how well a given flow solution approximates the flow of the usual polytropic gas, but also of how well it approximates the flow of an arbitrary barotropic fluid.

As illustrations of the use of the error expressions, they are applied to three basically different types of approximation to the flow of a polytropic gas through a cascade of typical compressor blading; namely, (1) the incompressible approximation, (2) the Prandtl-Glauert approximation, and (3) an approximation based on a linear relation between pressure and specific volume. The approximation based on the linear pressure-volume relation gave much the best agreement by satisfying the irrotationality and continuity conditions exactly and the momentum conditions within about 1 percent. Although it is shown that in the Prandtl-Glauert approximation, the four necessary conditions for the compressible flow are satisfied to the extent that terms higher than the first order in the perturbation velocity are negligible, the errors resulting from this linear approximation were so large for a typical compressor cascade that this approximation was no better than the incompressible approximation (maximum error expression, 19 percent). When both parallel and normal components of the perturbation velocity were considered in computing the resultant compressible velocity, however, appreciably better accuracy (maximum error expression, 7 percent) was obtained.

INTRODUCTION

The compressible flow along the blade surface must be accurately determined in designing blade rows for high loading and high subsonic Mach numbers in order to avoid excessive local velocities and to prevent flow separation. Because the solution for the two-dimensional potential flow of a compressible perfect fluid through a cascade of blades cannot be obtained analytically, some approximate method must be used. The methods of approximation consist in replacing the original partial differential equation by either a simpler partial differential equation or a difference equation and solving. The solution of the simpler equation may involve additional approximations, such as modified boundary conditions.

Some methods of approximation give the flow throughout the field and others permit the determination of the flow on the blade surface directly. For all methods, however, it is desirable to obtain high accuracy on the blade surface where the conditions are the most critical because there the velocity reaches its maximum value and its distribution along the blade surface determines the boundary-layer behavior. In general, the errors involved in the approximate methods are of two types: (1) errors arising from approximating the exact differential equation; and (2) errors arising in the solution of the simplified equation. Because the approximate solutions do not usually give in themselves a convenient means of checking the over-all accuracy obtained, an investigation was conducted at the NACA Lewis laboratory to obtain criterions for measuring and comparing the accuracy or consistency of the various approximate compressible-flow solutions. In a manner similar to that used by Weinig in reference 1 to obtain checks for the numerical accuracy of incompressible-flow solutions, four integral relations based on the necessary conditions of irrotationality and conservation of mass and momentum were obtained. Error expressions which give the deviation of an approximate compressible-flow solution from these necessary conditions are presented for measuring the accuracy of compressible-flow solutions. Because the expressions are given in a general form which permits using an arbitrary relation between pressure and density, they may be used to estimate how closely a given solution approximates the flow of any arbitrary barotropic fluid.

As an illustration of the application, the integral expressions are used to evaluate the relative accuracy of three different methods of approximating the usual compressible flow of a polytropic gas (a perfect gas with constant specific heat) through a cascade of blades. The three methods of approximation compared are: (1) the incompressible potential flow approximation, (2) the Prandtl-Glauert approximation as adapted to cascades by Woolard (reference 2), and (3) an approximation based on the linear pressure-volume relation as applied to cascades in reference 3.

THEORY

The relations for conservation of mass and momentum for steady two-dimensional compressible flow may be written

$$\oint_C (-\rho V_x \, dy + \rho V_y \, dx) = 0 \tag{1}$$

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$$\oint_C \left[p \, dx + \rho V_y(-V_x \, dy + V_y \, dx)\right] = 0 \tag{2}$$

and

$$\oint_C \left[-p \, dy + \rho V_x(-V_x \, dy + V_y \, dx)\right] = 0 \tag{3}$$

where the contour integrals are evaluated around any circuit C enclosing only fluid, that is, not enclosing any blades. (All symbols are defined in appendix A.) The irrotationality condition, which requires the circulation around the path C to be zero, may be written in the form

$$\oint_C (V_x \, dx + V_y \, dy) = 0 \tag{4}$$

Equations (1) to (4) would, of course, be true for either direction of integration along C, but this direction was taken as clockwise for the sake of definiteness.

If the contour integrals through a cascade of blades are evaluated along the contour C indicated by arrows along abcdefgha in figure 1, the portions of the integrals on the free sections of one stagnation streamline ha and bc cancel the portions on the free sections of the other stagnation streamline fg and de because the integrands are equal at corresponding points and the integrations are in opposite directions. The only remaining portions of the contour integrals are those along ab and ef, which give the integrals around the blade, and those along cd and gh, which can be readily evaluated in terms of the uniform upstream and downstream conditions. The contour integrals in equations (1) to (4) thus become

$$\oint_C (-\rho V_x \, dy + \rho V_y \, dx) = -\int_c^d \rho V_x \, dy - \int_g^h \rho V_x \, dy = (\rho V_x)_1 S - (\rho V_x)_2 S \tag{5}$$

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$$\oint_C \left[p \, dx + \rho V_y(-V_x \, dy + V_y \, dx)\right]$$

$$= \int_a^b p \, dx + \int_e^f p \, dx - \int_c^d \rho V_y V_x \, dy - \int_g^h \rho V_y V_x \, dy$$

$$= \oint_B p \, dx + (\rho V_y V_x)_1 S - (\rho V_y V_x)_2 S \tag{6}$$

$$\oint_C \left[-p \, dy + \rho V_x(-V_x \, dy + V_y \, dx)\right]$$

$$= -\int_a^b p \, dy - \int_e^f p \, dy - \int_c^d p \, dy - \int_g^h p \, dy - \int_c^d \rho V_x^2 \, dy - \int_g^h \rho V_x^2 \, dy$$

$$= -\oint_B p \, dy + p_1 S - p_2 S + (\rho V_x^2)_1 S - (\rho V_x^2)_2 S \tag{7}$$

$$\oint_C (V_x \, dx + V_y \, dy) = \int_a^b V_s \, ds + \int_e^f V_s \, ds + \int_c^d V_y \, dy + \int_g^h V_y \, dy$$

$$= \oint_B V_s \, ds - V_{1,y} S + V_{2,y} S \tag{8}$$

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where B indicates the integration path around the blade contour (counterclockwise) and V_s is taken as the component of the velocity in the direction of integration or increasing s.

For any potential flow in which the density is a known function of the pressure, the density and the pressure can be expressed as a function of the velocity by means of Bernoulli's equation

$$\int_{p_T}^{p} \frac{dp}{\rho(p)} + \frac{V^2}{2} = 0 \tag{9}$$

Consequently, if the approximate compressible-flow solution is given in terms of the velocity distribution, all the separate portions of the contour integrals can be evaluated and the magnitude of the deviation from the necessary conditions (1) to (4) determined. These deviations are expressed as follows: the percentage error in mass flow

$$\delta_M = \frac{(\rho V_x)_2 - (\rho V_x)_1}{(\rho V_x)_1} \times 100 \tag{10}$$

the error in blade force components as percent of the resultant force

$$\delta_y = \frac{\oint_B p\, dx - \left[(\rho V_y V_x)_2 - (\rho V_y V_x)_1\right] s}{s\sqrt{\left[(\rho V_y V_x)_2 - (\rho V_y V_x)_1\right]^2 + \left[p_2 - p_1 + (\rho V_x^{\,2})_2 - (\rho V_x^{\,2})_1\right]^2}} \times 100 \tag{11}$$

and

$$\delta_x = \frac{-\oint_B p\, dy - \left[p_2 - p_1 + (\rho V_x^{\,2})_2 - (\rho V_x^{\,2})_1\right] s}{s\sqrt{\left[p_2 - p_1 + (\rho V_x^{\,2})_2 - (\rho V_x^{\,2})_1\right]^2 + \left[(\rho V_y V_x)_2 - (\rho V_y V_x)_1\right]^2}} \times 100 \tag{12}$$

and the percentage error in circulation

$$\delta_T = \frac{\oint_B V_s \, ds - (V_{1,y} - V_{2,y})S}{(V_{1,y} - V_{2,y})S} \times 100 \tag{13}$$

EVALUATION OF THREE TYPES OF COMPRESSIBLE-FLOW APPROXIMATION USING ERROR EXPRESSIONS

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Three different methods were used for approximating the compressible flow through a cascade of typical compressor blades to serve as an illustration of practical application of the error expressions (10) to (13) and their relative accuracy was estimated by the use of these four error expressions.

Incompressible flow as approximation to compressible flow. - For many applications, incompressible-flow solutions have proved to be satisfactory approximations to the desired compressible-flow solutions. Because of this observation and because of the relative simplicity of incompressible flow, until recently practically all the extensive literature on computed flow through cascades has been devoted to incompressible flow (most of the literature prior to 1949 is summarized in reference 4). The first approximation method investigated is therefore an incompressible approximation. In this approximation, the dimensionless compressible velocity V_c/a_T is assumed to be equal everywhere to the velocity V_i for some known incompressible-flow solution. Inasmuch as the velocity for any incompressible flow may be altered by a scale factor, a family of compressible-flow approximations can be obtained from any known incompressible solution. The accuracy of the approximation decreases as the scale of the dimensionless compressible velocity V_c/a_T, and hence the Mach number M, is increased.

In order that the example illustrating the application of the error expressions might be of practical interest, the cascade and the scale of the velocities were chosen to be representative of axial-flow compressors. The incompressible flow was obtained by a numerical solution using the theoretically exact inverse method of reference 5 wherein the blade shape is computed from the prescribed velocity distribution. The velocity distribution chosen and the resulting cascade geometry for this example are shown in figures 2 and 3, respectively. In this example, as well as in those following in this report, the total blade arc length was taken as 2π. The maximum dimensionless compressible velocity on the blade obtained by this approximation is 0.7558, which corresponds to a Mach number of 0.8031.

In order to obtain measures of the accuracy of this approximation to the usual compressible flow of a polytropic gas, the error expressions (10) to (13) were computed using the relations for the adiabatic flow

$$\frac{\rho}{\rho_T} = \left(1 - \frac{\gamma-1}{2}\,\frac{V_c{}^2}{a_T{}^2}\right)^{\frac{1}{\gamma-1}} \tag{14}$$

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and

$$\frac{p}{p_T} = \left(1 - \frac{\gamma-1}{2}\,\frac{V_c{}^2}{a_T{}^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{15}$$

These relations may be readily obtained from the general Bernoulli equation (9) using the adiabatic relation for a polytropic gas

$$\frac{p}{p_T} = \left(\frac{\rho}{\rho_T}\right)^{\gamma} \tag{16}$$

Because the blade-profile velocities of the incompressible flow on which this approximation was based were computed at equally spaced points on the unit circle into which the blade profile is transformed, the integrals also were expressed in terms of the unit circle in order that Simpson's rule might readily be applied. Thus

$$\left.\begin{aligned}
\oint_B p\,dx &= \int_0^{2\pi} p(\theta)\,\frac{dx}{d\theta}\,d\theta \\
\oint_B p\,dy &= \int_0^{2\pi} p(\theta)\,\frac{dy}{d\theta}\,d\theta \\
\oint_B V_s\,ds &= \int_0^{2\pi} V_s(\theta)\,\frac{ds}{d\theta}\,d\theta
\end{aligned}\right\} \tag{17}$$

Although the incompressible solution on which this approximation to compressible flow was based was obtained by a theoretically exact method, there still remain numerical inaccuracies due to rounded-off values, approximation methods for numerical integration, etc. It was therefore desirable to have some measure of the accuracy of the incompressible solution itself. The accuracy of the incompressible solution was measured by the same expressions (10) to (13) as for the compressible approximation but with constant density and with the pressures obtained from Bernoulli's equation for incompressible flow

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$$p = p_T - \frac{\rho V^2}{2} \tag{18}$$

The numerical values of the error expressions obtained for the incompressible solution and the incompressible approximation to a compressible flow with $\gamma = 1.4$ are shown in the following table:

Method	Error (percent)			
	δ_M	δ_y	δ_x	δ_Γ
Incompressible flow	0.00	-0.09	- 0.06	-0.07
Incompressible approximation to compressible flow	6.72	2.05	-14.14	-0.07

A comparison of the relative errors in this table indicates that the numerical inaccuracies in obtaining the incompressible solution are quite insignificant compared with the inaccuracies involved in this approximation to compressible flow. The large percentage error in the x-component of force indicates that the incompressible flow is a poor approximation to compressible flow through the cascade at this Mach number.

Prandtl-Glauert approximation. - The incompressible solution was next used to obtain a compressible-flow solution based on the Prandtl-Glauert approximation as applied to cascades by Woolard (reference 2). This approximation is based on the linearized equation for the perturbation-velocity potential and uses only linear terms in the perturbation-velocity components. Woolard considers the vector mean of the upstream and downstream incompressible velocities $V_{m,i}$ as the reference velocity and deviation from this velocity as the perturbation velocity. If the transformation

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$$\left.\begin{array}{c} X_i = X_c \qquad Y_i = \Omega Y_c \qquad \Phi_i = k\Phi_c \\ V_{m,i} = V_{m,c} = V_m \end{array}\right\} \tag{19}$$

is used where $\Omega = \sqrt{1-M_m^2}$ and X is the coordinate parallel to, and Y, the coordinate normal to the vector mean velocity, a linear approximation to a compressible flow through a cascade is obtained from an incompressible flow through a related cascade. The compressible cascade has the same chord and spacing in the X-direction S_X as the incompressible cascade, but the spacing in the Y-direction S_Y is increased by the factor $1/\Omega$. The ratio of the compressible to the incompressible values of blade thickness, camber, and air turning angle will be equal (within the linear approximation) to Ω/k. It is shown in appendix B that the four integral expressions (1) to (4) are satisfied within the linear terms in the perturbation-velocity components for the preceding interpretation of the Prandtl-Glauert rule.

For the linear approximation to be reasonably accurate, however, it is necessary that the angle of attack, turning angle, blade camber, blade thickness, blade surface slope, etc. be sufficiently small that the compressible and incompressible perturbation velocities be very small compared with the mean velocity V_m. In most practical cases these conditions will not be strictly satisfied everywhere and, therefore, expressions (10) to (13) serve as convenient measures of resulting mean errors in any particular example. As an illustration of this use, the preceding interpretation of the Prandtl-Glauert rule for the case of $k = \Omega$ was applied to the incompressible-flow solution discussed previously, and the error expressions were determined for this approximation to compressible flow. In this case $(k = \Omega)$, the blade section and the turning angle are the same (within the linear approximation) as for the incompressible flow and for the incompressible approximation, but the blade spacing S_Y is increased by the factor $1/\Omega$. Because of this change in S_Y, the flow angles with respect to the normal to the cascade axis are also changed. The relation between the mean flow angles is given by

$$\tan \lambda_{m,c} = \Omega \tan \lambda_{m,i}$$

The mean Mach number was taken the same as for the incompressible approximation to compressible flow in order that the error expressions might be used to measure the relative accuracy of the two approximations under comparable conditions. The cascade geometry for the compressible flow is shown in figure 4. The blade spacing and orientation for the incompressible cascade are also shown for comparison.

Inasmuch as the Prandtl-Glauert theory is based on the assumption that the perturbation velocity is sufficiently small that powers higher than the first in the perturbation-velocity components may be neglected throughout, the use of other than linearized expressions in these components in computing the resultant velocity might seem unjustifiable. For most practical cases, however, quite different results are obtained if the linearized expression

$$V = V_m + u \tag{20}$$

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is used rather than the exact expression

$$V = \sqrt{(V_m + u)^2 + v^2} \tag{21}$$

The compressible velocity distribution has been computed for the cascade example of figure 4 by both equation (20) and equation (21) using the Prandtl-Glauert rule for obtaining the compressible perturbation-velocity components (with $k = \Omega$).

$$u_c = u_i/\Omega \qquad v_c = v_i$$

The resulting velocity distributions are shown in figure 5. The appreciable differences over a large part of the upper surface are a result of the departures of the direction of the tangent to the upper surface from a direction parallel to the mean velocity. (See fig. 4.) Both velocity distributions have been faired over a small region near the nose and the tail of the blade to give a single stagnation point in these regions, but this fairing was over such a small region as to be hardly detectable on the scale of figure 5. Because of the appreciable difference between the two velocity distributions, a corresponding difference in the value of the error expressions might be expected. The values of the error expressions using equations (14) and (15) with $\gamma = 1.4$ and based both on the exact equations for the resultant velocity and the velocity components and on the corresponding equations involving only the linear terms in the perturbation-velocity components are given in the following table:

Method	Error (percent)			
	δ_M	δ_y	δ_x	δ_Γ
Linearized velocity relation	-0.11	-15.34	-12.23	-18.91
Exact velocity relation	0.02	- 6.63	- 3.72	- 4.45

The Prandtl-Glauert rule, with the use of the linearized relation for the velocity, gives error expressions, of which the maximum for this example is of about the same order of magnitude as for the incompressible approximation to compressible flow. Thus, apparently little is to be gained in the accuracy of the velocity distribution by the use of the Prandtl-Glauert rule in this form when applied to typical compressor blading of moderate camber. The results should be even less accurate for typical turbine blading in which the camber is generally much larger.

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The reason for this poor performance is clarified by the comparison of the incompressible velocity distribution based on the same linearized approximation with the original incompressible velocity distribution in figure 6. The difference is of the same order of magnitude as that between the two interpretations of the Prandtl-Glauert rule shown in figure 5. The values of the error expressions for the linearized approximation to the incompressible flow, as compared with those for the theoretically exact incompressible flow (which represent computational inaccuracies) are as follows:

Method	Error (percent)			
	δ_M	δ_y	δ_x	δ_Γ
Linearized velocity relation	0.26	-9.77	-11.92	-17.90
Exact velocity relation	0.00	-0.09	- 0.06	- 0.07

The preceding table together with figures 5 and 6 indicates that the errors due to linearization are quite large. In fact, these errors are of the same order of magnitude as the compressibility correction for this cascade example as is seen by comparing the linearization error shown in figure 6 with the compressibility correction shown in figure 7. The considerably better results obtained from the Prandtl-Glauert rule when using the exact relations than when using the linearized relations is thus a result of elimination of the direct error due to linearization for the incompressible flow. There remains, however, the error due to linearization in computing the compressibility correction. Whenever the direct errors due to linearization are as large as or larger than the compressibility correction, more accurate results may be expected if exact relations for computing the resultant velocities and velocity components from the perturbation velocities are used.

Approximation based on linear pressure-volume relation. - As a third type of approximation method to be investigated, the inverse method of reference 3 based on the use of a linear relation between pressure and specific volume was used. In this method, a proportionality constant is determined between the dimensionless compressible velocities (ratios of fluid velocity to stagnation velocity of sound) for the approximated compressible flow of a polytropic gas and a theoretically exact compressible flow of an ideal fluid with a linear relation between pressure and specific volume. The proportionality constant is determined in such a way that the continuity equation is satisfied for the approximated flow, that is, $\delta_M = 0$. Because the flow with the linear pressure-volume relation is irrotational, the approximated flow will also be irrotational and thus satisfy the circulation condition (4), that is, $\delta_\Gamma = 0$, within the numerical accuracy of the solution. The momentum conditions (2) and (3) will not, however, be exactly satisfied and the values of δ_y and δ_x will therefore serve as a measure of the accuracy of the approximation. 2251

In order that this approximation method might reasonably be compared with the previously considered methods, it was desirable to obtain a cascade geometry and flow conditions similar to the other approximations. Because this approximation as well as the incompressible-flow approximation was obtained from inverse design methods, it was impossible to obtain exactly the same blade geometry. Furthermore, the incompressible approximation and the Prandtl-Glauert approximation are for somewhat different configurations because of the different values of S_Y. The Mach number based on the vector mean of the upstream and downstream velocities was chosen the same for all the different approximations, and the velocity distribution for the present approximation (fig. 8) was chosen to be very nearly the same as for the example of the Prandtl-Glauert approximation in which the exact velocity relation (fig. 5) was used, but blade shape, blade spacing, and flow angles are somewhat different (figs. 4 and 9). However, the differences between the different approximation examples are small enough to make relative values of the error expression significant in comparing the relative accuracy of the different approximation methods.

As in the case of the incompressible-flow solution, it is desirable to have some measure of the numerical inaccuracies of a particular solution independent of the errors in approximating the desired flow of the usual polytropic gas $(\gamma = 1.4)$. The numerical inaccuracies in obtaining the flow with a linear pressure-volume relation $(\gamma = -1)$ can readily be obtained from the general error expressions (10) to (13) by using the proper equations for dimensionless pressure and density for a gas with linear pressure-volume relation

$$p = A - \sqrt{1 + q^2} \tag{22}$$

$$\rho = \frac{1}{\sqrt{1 + q^2}} \tag{23}$$

where A is an arbitrary constant and q is the dimensionless velocity (ratio of velocity to stagnation velocity of sound) for this gas. The constant A does not affect the values of any of the error expressions for closed blade profiles and was therefore taken as zero.

For the present cascade example, the error expressions have been calculated both for the linearized pressure-volume flow $(\gamma = -1)$ by using equations (22) and (23) and for a polytropic gas with $\gamma = 1.4$ by using equations (14) and (15); the results are shown in the following table:

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Method	Error (percent)			
	δ_M	δ_y	δ_x	δ_Γ
Linearized pressure-volume flow	0.00	0.47	0.46	0.00
Approximated flow for $\gamma = 1.4$	0.00	-0.33	-0.99	0.00

Because this method is theoretically exact for the linearized pressure-volume flow, the error expressions for this case are measures of the numerical inaccuracies due to rounded-off values, integration approximations, etc. The error expressions for the other case $(\gamma = 1.4)$ are a measure of the combined error due to numerical inaccuracies and the inherent error resulting from assuming the velocity V_c for this flow to be proportional to the velocity for the linearized pressure-volume flow $(\gamma = -1)$. The errors for the approximated flow are apparently only slightly greater than the numerical inaccuracies in the calculation. The error expressions have also been calculated for all the examples given in reference 3 and for all cases were below 1 percent. Thus, apparently this method is quite accurate for a wide variety of examples and is not limited to flow with small perturbations from the mean velocity, as is the case with the Prandtl-Glauert approximation.

For convenient comparison, the error expressions for the examples of the different approximations to flow with $\gamma = 1.4$ previously presented are summarized as follows:

Method	Error (percent)			
	δ_M	δ_y	δ_x	δ_Γ
Incompressible approximation	6.72	2.05	-14.14	- 0.07
Prandtl-Glauert, linear velocity relation	-0.11	-15.34	-12.23	-18.91
Prandtl-Glauert, exact velocity relation	0.02	- 6.63	- 3.72	- 4.45
Approximation based on linear pressure-volume relation	0.00	- 0.33	- 0.99	0.00

DISCUSSION

It should be emphasized that the error expressions measure how closely certain necessary conditions, representative of particularly significant features of the flow, are satisfied; but even if they are all zero, the solution may still be inaccurate. An infinite number of conditions are required for a solution to be exact; namely, that it agree with an exact solution everywhere. Inasmuch as the error expressions are based on integrals, they are essentially particular methods of averaging the errors of a solution. In certain pathological cases, the errors may cancel each other in the integration, giving a null value for the error expressions. If they are used with discretion, however, the error expressions furnish valuable checks on the accuracy of compressible-flow solutions.

Although the expressions may be used in checking any type of approximate solution, they are particularly valuable in checking solutions which give the velocity or pressure distribution on the blade surface directly, as there are often no other convenient checks available in this case. For relaxation solutions or other solutions which give the flow throughout the field, the residual of the difference equation approximating the differential equation furnishes other indications of the error, but does not indicate the error involved in approximating the differential equation. The error expressions thus furnish valuable supplementary checks in these cases.

For convenience of discussion, it has previously been assumed that the approximate flow solution is given in the form of the dimensionless velocity distribution, but this is unnecessary. The solution may be specified in any one of a variety of ways, such as the upstream and downstream Mach numbers and a pressure-coefficient distribution around the blade. The error expressions would then be calculated using the relations for density, pressure, and velocity in terms of the specified variables which are exact for the fluid which it is desired to approximate.

Lewis Flight Propulsion Laboratory,
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APPENDIX A

SYMBOLS

The following symbols are used in this report:

A arbitrary constant

a velocity of sound

B integration path along blade contour

C integration path enclosing only fluid

k arbitrary constant

M Mach number

p pressure

q ratio of velocity to stagnation velocity of sound for gas with linear pressure-volume relation

S cascade spacing

s arc length

u perturbation-velocity component parallel to mean velocity

V velocity

v perturbation-velocity component normal to mean velocity

X,Y rectangular coordinates, parallel and normal, respectively, to mean velocity

ΔY_B range of integration of Y on blade

x,y rectangular coordinates normal and parallel, respectively, to cascade axis

γ ratio of specific heats

δ_M error in mass flow, percent (equation (10))

δ_x error in x-component of blade force, percent of resultant force (equation (12))

δ_y error in y-component of blade force, percent of resultant force (equation (11))

δ_Γ error in circulation, percent (equation (13))

θ circle angle

λ angle between velocity vector and normal to cascade

ρ density

σ solidity

Φ perturbation-velocity potential

Ω constant, $\sqrt{1 - M_m{}^2}$

Subscripts:

1 far upstream of cascade

2 far downstream of cascade

B along blade contour

c compressible

i incompressible

m values with reference to vector mean of upstream and downstream velocity

max maximum value

s component in s-direction

T total or stagnation conditions

X component in X-direction

x component in x-direction

Y component in Y-direction

y component in y-direction

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APPENDIX B

PROOF THAT EQUATIONS (1) TO (4) ARE SATISFIED WITHIN LINEAR APPROXIMATION FOR PRANDTL-GLAUERT RULE

It will be shown that equations (1) to (4), which are evaluated along contour C of figure 1, are satisfied for the Prandtl-Glauert approximation within linear terms of the perturbation-velocity components; that is, they are satisfied to the approximation that higher-order terms in the compressible and incompressible perturbation-velocity components may be neglected in comparison with the linear terms in computing pressures, densities, and velocities for the compressible and incompressible flows. Equations which are accurate only within linear terms in the perturbation velocity will be indicated by the sign $\approx$.

In establishing that these equations are satisfied within this linear approximation, the equations are for convenience expressed in terms of the coordinates X,Y parallel and normal, respectively, to the mean velocity rather than in terms of the coordinates x,y normal and parallel, respectively, to the cascade axis. The expressions in the new coordinates may readily be obtained from the general equations (1) to (4) in the same manner as equations (5) to (8) were obtained. The resulting expressions are

$$-\left[(\rho V_X)_2 - (\rho V_X)_1\right]S_Y - \left[(\rho V_Y)_2 - (\rho V_Y)_1\right]S_X = 0 \qquad \text{(B1)}$$

$$\oint_B p\,dX - (p_2 - p_1)S_X - \left[(\rho V_X V_Y)_2 - (\rho V_X V_Y)_1\right]S_Y - \left[(\rho V_Y{}^2)_2 - (\rho V_Y{}^2)_1\right]S_X = 0 \qquad \text{(B2)}$$

$$-\oint_B p\,dY - (p_2 - p_1)S_Y - \left[(\rho V_X{}^2)_2 - (\rho V_X{}^2)_1\right]S_Y - \left[(\rho V_Y V_X)_2 - (\rho V_Y V_X)_1\right]S_X = 0 \qquad \text{(B3)}$$

$$\oint_B V_s\,ds + (V_{Y,2}-V_{Y,1})S_Y - (V_{X,2}-V_{X,1})S_X = 0 \tag{B4}$$

In order to check whether these equations are satisfied within the linear terms in the perturbation-velocity components u,v defined as

$$u = V_X - V_m$$

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and

$$v = V_Y$$

the equation will now be expressed in terms of these pertubation-velocity components in which only linear terms will be considered. The values of V_m, p_m, ρ_m, S_Y, S, and chord are taken as standard or zeroth order. The perturbation-velocity components u and v for both the compressible and incompressible flow are assumed to be of the first order. Camber, thickness, range of Y on a blade, and turning angle must then be of the first order because of the relation along any streamline

$$\frac{dY}{dX} = \frac{v}{V_m+u} \approx \frac{v}{V_m}$$

The linearized expressions are obtained as follows:

$$\left.\begin{aligned}
\frac{p}{p_m} &= \left[\frac{1-\frac{\gamma-1}{2}\frac{V^2}{a_T^2}}{1-\frac{\gamma-1}{2}\frac{V_m^2}{a_T^2}}\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{1-\frac{\gamma-1}{2}\frac{V_m^2}{a_T^2}\left(1+\frac{u}{V_m}\right)^2-\frac{\gamma-1}{2}\frac{v^2}{a_T^2}}{1-\frac{\gamma-1}{2}\frac{V_m^2}{a_T^2}}\right]^{\frac{\gamma}{\gamma-1}} \\
&\approx \left[1-\frac{\frac{(\gamma-1)V_m^2}{a_T^2}\frac{u}{V_m}}{1-\frac{\gamma-1}{2}\frac{V_m^2}{a_T^2}}\right]^{\frac{\gamma}{\gamma-1}} \approx \left[1-\frac{(\gamma-1)V_m^2}{a_m^2}\frac{u}{V_m}\right]^{\frac{\gamma}{\gamma-1}} \\
&\approx 1-\gamma M_m^2\frac{u}{V_m} \approx 1-\frac{\rho_m V_m^2}{p_m}\frac{u}{V_m}
\end{aligned}\right\} \tag{B5}$$

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Hence

$$p - p_m \approx -\rho_m V_m^2 \frac{u}{V_m} \tag{B6}$$

Similarly

$$\frac{\rho}{\rho_m} \approx \left[1 - (\gamma-1)M_m^2 \frac{u}{V_m}\right]^{\frac{1}{\gamma-1}} \approx 1 - M_m^2 \frac{u}{V_m} \approx 1 - \frac{\rho_m V_m^2}{\gamma p_m} \frac{u}{V_m} \tag{B7}$$

Therefore

$$\rho V_X \approx \rho_m V_m \left(1 - M_m^2 \frac{u}{V_m}\right)\left(1+\frac{u}{V_m}\right) \approx \rho_m V_m \left[1 + (1-M_m^2) \frac{u}{V_m}\right] \tag{B8}$$

$$\rho V_Y \approx \rho_m V_m \left(1 - M_m^2 \frac{u}{V_m}\right) \frac{v}{V_m} \approx \rho_m V_m \frac{v}{V_m} \tag{B9}$$

$$\rho V_X^2 \approx \rho_m V_m^2 \left(1 - M_m^2 \frac{u}{V_m}\right)\left(1+\frac{2u}{V_m}\right) \approx \rho_m V_m^2 \left[1 + (2-M_m^2) \frac{u}{V_m}\right] \tag{B10}$$

$$\rho V_X V_Y \approx \rho_m V_m^2 \frac{v}{V_m} \tag{B11}$$

$$\rho V_Y^2 \approx \rho_m V_m^2 \left(\frac{v}{V_m}\right)^2 \approx 0 \tag{B12}$$

$$\left[\oint_B p \, dY\right] \approx \left[\rho_m V_m^2 \oint_B \frac{u}{V_m} \, dY\right] \leq \left[\rho_m V_m^2 \frac{u_{max}}{V_m} \Delta Y_B\right]$$

where ΔY_B indicates range of integration of Y on the blade. Since both ΔY_B and u_{max} are of the first order, the integral is equal to zero to the first order. This merely means that the blade force is normal to the mean velocity within the linear approximation, a condition which is exactly satisfied for incompressible flow. If these values are substituted in equations (B1) to (B4), the linearized form of each of these equations for compressible flow results.

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$$\rho_m V_m \left[-\left(1-M_m^2\right)\left(\frac{u_2-u_1}{V_m}\right)_c S_{Y,c} - \left(\frac{v_2-v_1}{V_m}\right)_c S_{X,c} \right] \approx 0 \tag{B13}$$

$$\rho_m V_m^2 \left[-\oint_{B,c} \left(\frac{u}{V_m}\right)_c dX_{B,c} + \left(\frac{u_2-u_1}{V_m}\right)_c S_{X,c} - \left(\frac{v_2-v_1}{V_m}\right)_c S_{Y,c} \right] \approx 0 \tag{B14}$$

$$\rho_m V_m^2 \left[-\left(1-M_m^2\right)\left(\frac{u_2-u_1}{V_m}\right)_c S_{Y,c} - \left(\frac{v_2-v_1}{V_m}\right)_c S_{X,c} \right] \approx 0 \tag{B15}$$

$$V_m \left[\oint_{B,c} \left(\frac{u}{V_m}\right)_c dX_{B,c} - \left(\frac{u_2-u_1}{V_m}\right)_c S_{X,c} + \left(\frac{v_2-v_1}{V_m}\right)_c S_{Y,c} \right] \approx 0 \tag{B16}$$

Because equations (B13) and (B15) are identical and equations (B14) and (B16) are identical, except for factors involving ρ_m and V_m, which are of standard order, the four independent conditions for the general case (equations (B1) to (B4)) are reduced to two independent conditions for the linear approximation.

In proving that these conditions are satisfied for solutions obtained by the Prandtl-Glauert rule, they are expressed as equivalent conditions on the incompressible flow by the Prandtl-Glauert transformation and it is then proved that these resulting conditions are satisfied within the linear approximation for the exact solution of the incompressible flow.

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From the basic Prandtl-Glauert relations (equation (19)), the following additional relations between the corresponding compressible and incompressible flows are obtained (reference 2):

$$S_{X,c} = S_{X,i} \qquad S_{Y,c} = \frac{1}{\Omega} S_{Y,i} \tag{B17}$$

$$u_c = \frac{1}{k} u_i \qquad v_c = \frac{\Omega}{k} v_i \tag{B18}$$

The substitution of these values in the two independent equations (B13) and (B14) gives the conditions on the compressible flow expressed in terms of the corresponding incompressible flow as follows:

$$\rho_m V_m \frac{\Omega}{k} \left[-\left(\frac{u_2-u_1}{V_m}\right)_i S_{Y,i} - \left(\frac{v_2-v_1}{V_m}\right)_i S_{X,i} \right] \approx 0 \tag{B19}$$

$$\rho_m V_m^2 \frac{1}{k} \left[\oint_{B,i} \left(\frac{u}{V_m}\right)_i dX_{B,i} - \left(\frac{u_2-u_1}{V_m}\right)_i S_{X,i} + \left(\frac{v_2-v_1}{V_m}\right)_i S_{Y,i} \right] \approx 0 \tag{B20}$$

The incompressible flow on which the Prandtl-Glauert approximation is based must satisfy the conditions (B1) to (B4) with ρ constant and p given by equation (18). As in the case of the compressible flow, the linear approximation to these conditions gives only two independent conditions, which may be most easily obtained by considering these conditions as the limiting case of a compressible flow as the Mach number approaches zero in equations (B13) and (B14). Thus

$$\rho_m V_m \left[-\left(\frac{u_2-u_1}{V_m}\right)_i S_{Y,i} - \left(\frac{v_2-v_1}{V_m}\right)_i S_{X,i} \right] \approx 0 \tag{B21}$$

$$\rho_m V_m^2 \left[\oint_{B,i} \left(\frac{u}{V_m}\right)_i dX_{B,i} - \left(\frac{u_2-u_1}{V_m}\right)_i S_{X,i} + \left(\frac{v_2-v_1}{V_m}\right)_i S_{Y,i} \right] \approx 0 \tag{B22}$$

Equation (B19) differs from equation (B21) by the factor Ω/k and equation (B20) differs from equation (B22) by the factor $1/k$. Thus, if Ω/k and $1/k$ are not large, the conditions on the compressible-flow equations (B19) and (B20) obtained from the Prandtl-Glauert rule are satisfied to the same order as the linearized conditions on the corresponding incompressible flow. According to equations (B18), this condition is equivalent to the condition that the compressible perturbation-velocity components u_c and v_c are of the same order of magnitude as the corresponding components for the incompressible flow.

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For the linearized form of the necessary conditions on the incompressible flow to be satisfied within a given accuracy, the incompressible perturbation-velocity components must be sufficiently small compared with V_m and, consequently, blade-surface slope, blade camber, blade thickness, angle of attack, turning angle, etc. of the incompressible cascade must be sufficiently small. In order to guarantee that the corresponding compressible flow satisfy the necessary condition with similar accuracy, the compressible perturbation velocities u_c and v_c must also be sufficiently small. If, in a given incompressible cascade, the values of u_i and v_i are equal to the maximum allowable values for the required accuracy and it is required that values of u_c and v_c for the corresponding compressible cascade not exceed these values, it is necessary that $k \geq 1$. Inasmuch as the ratio of the compressible to the incompressible values of blade camber, blade thickness, angle of attack, turning angle, etc. are equal to Ω/k, the ratio of these quantities must not be greater than $\sqrt{1-M_m^2}$ and, consequently, blade thickness, turning angle, etc. for the compressible flow must be extremely small if M_m is near 1 in order for the Prandtl-Glauert approximation to be reasonably accurate.

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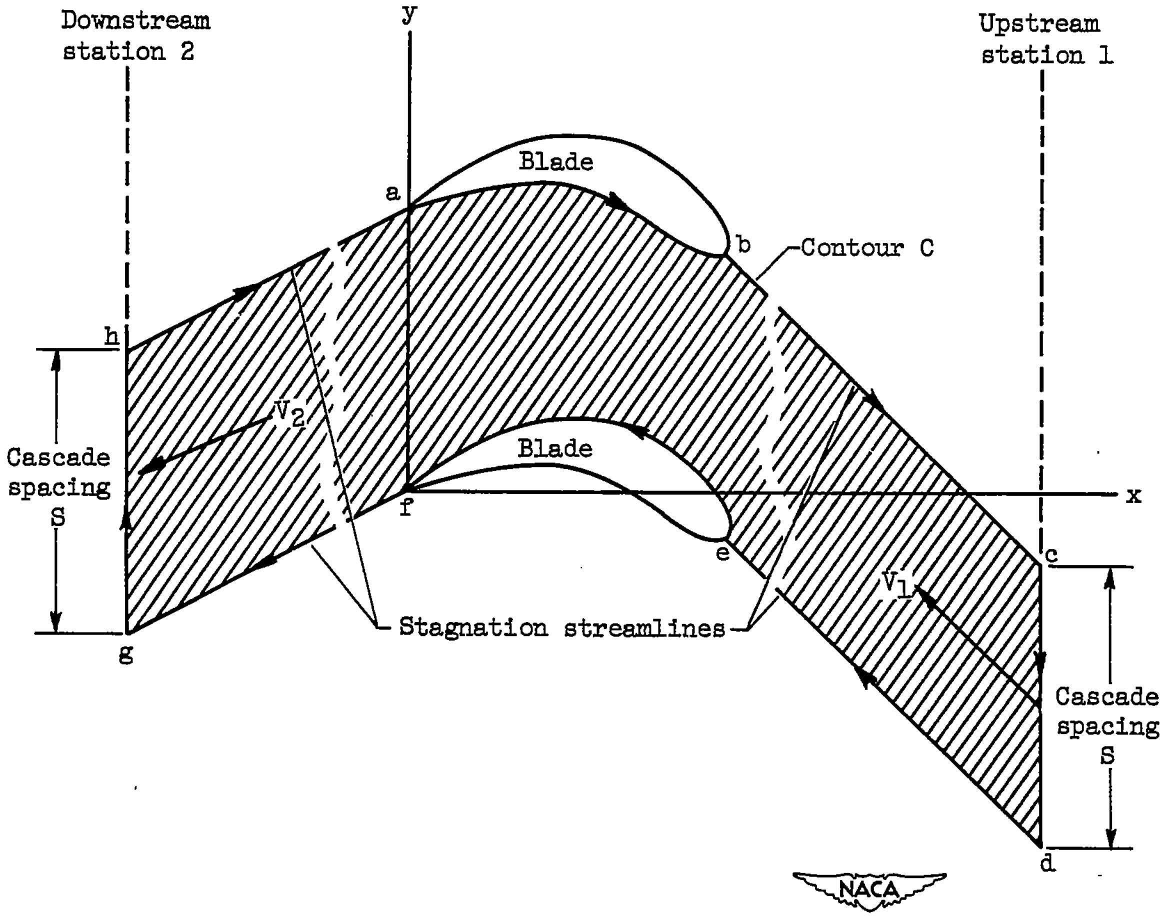

Figure 1. - Path of integration through cascade of blades used in general analysis.

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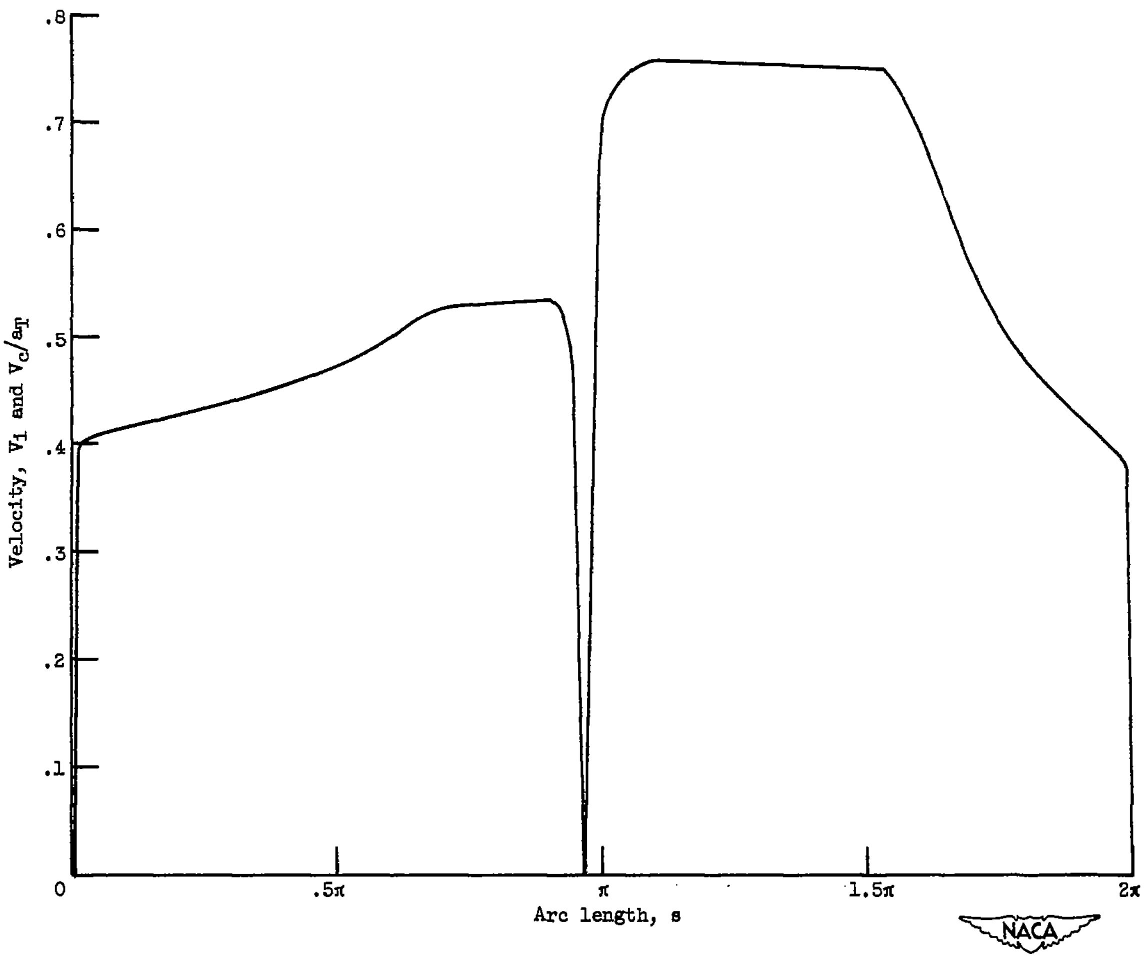

Figure 2. - Velocity distribution for incompressible flow and incompressible approximation.

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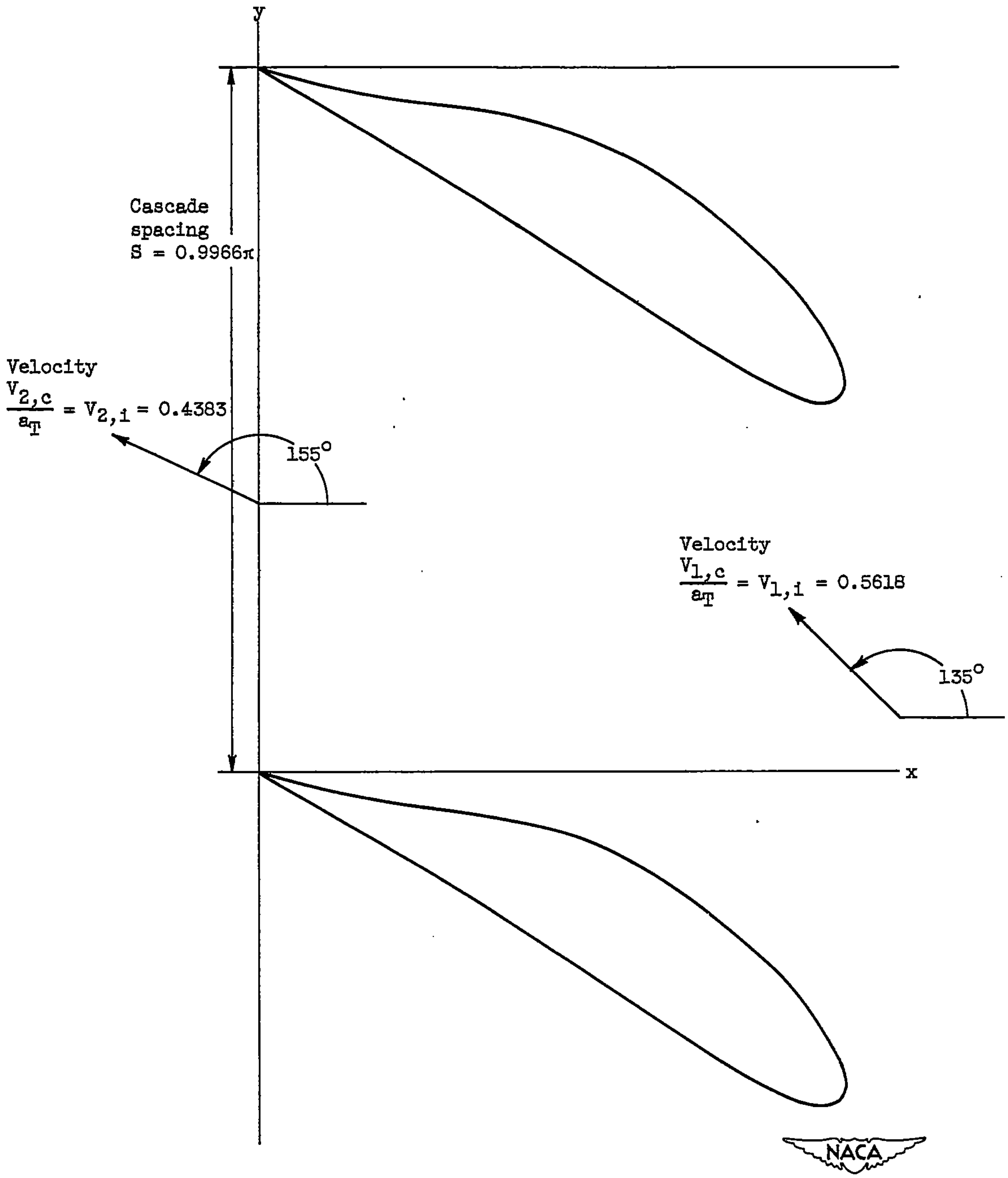

Figure 3. - Cascade geometry for incompressible flow and incompressible approximation. Mean Mach number, 0.5050; incompressible and compressible solidity, 0.9571.

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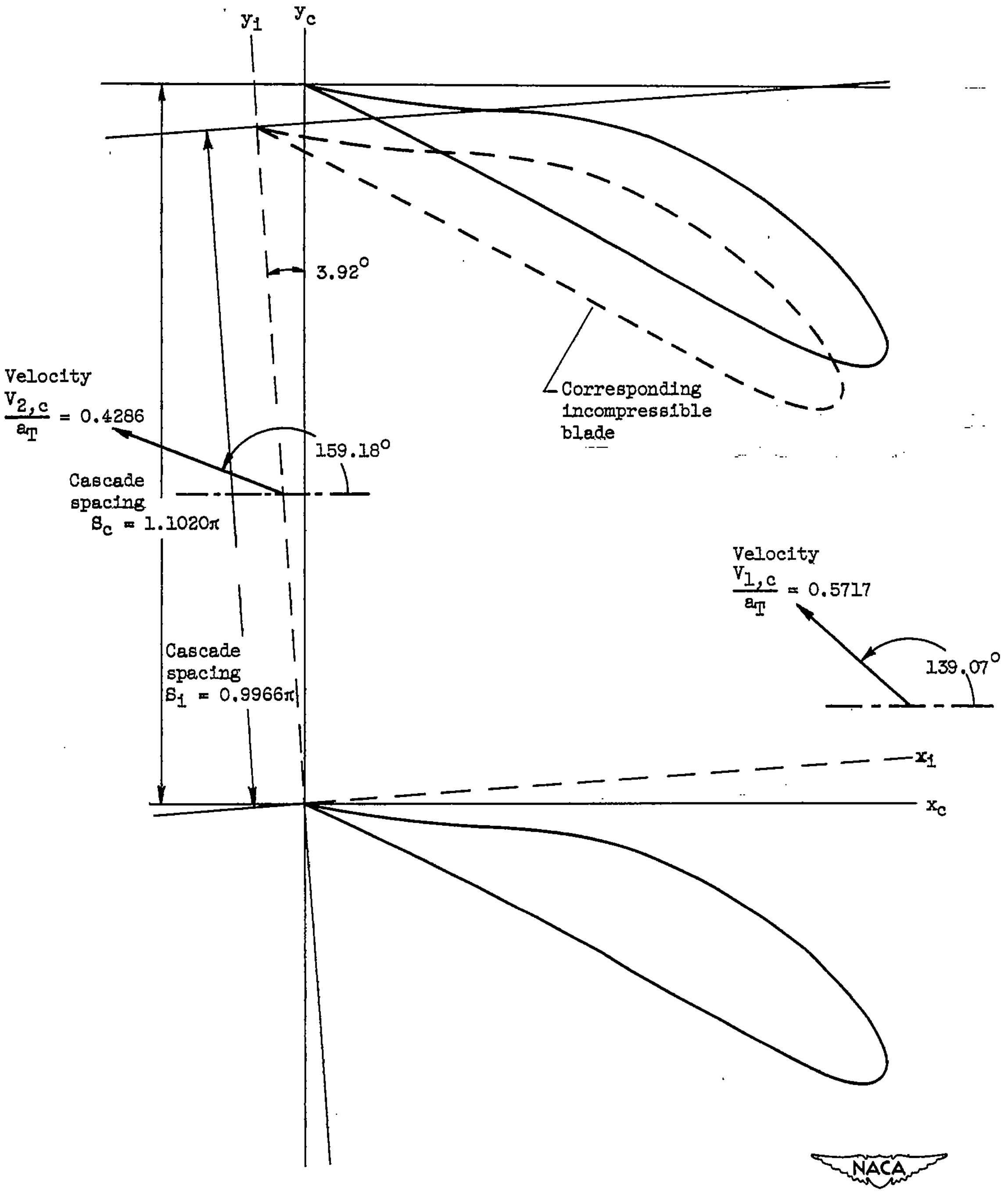

Figure 4. - Cascade geometry for Prandtl-Glauert approximation to compressible flow. Mean Mach number, 0.5050; compressible solidity, 0.8655; velocity $V = \sqrt{(V_m + u)^2 + v^2}$. Incompressible cascade geometry also shown for comparison.

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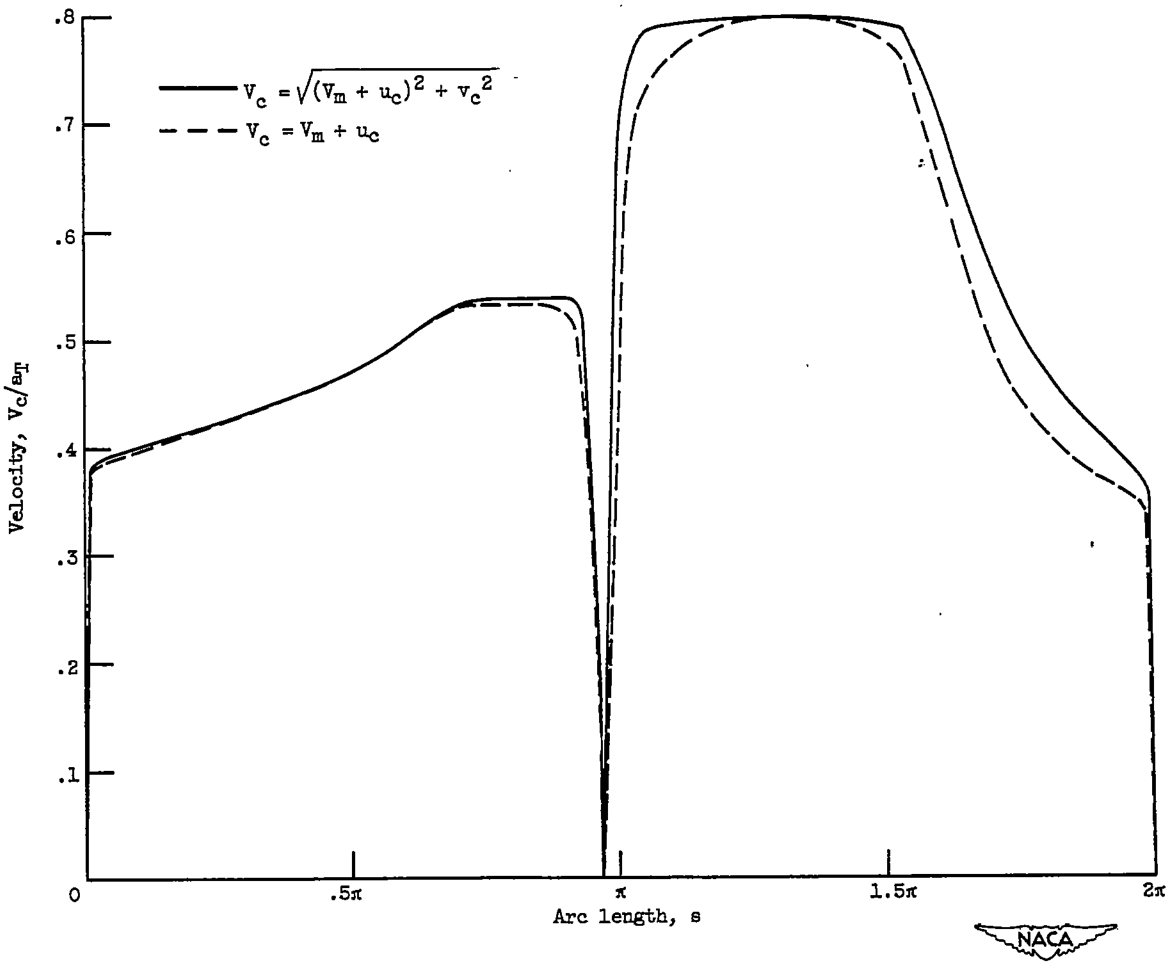

Figure 5. - Velocity distributions for Prandtl-Glauert approximations.

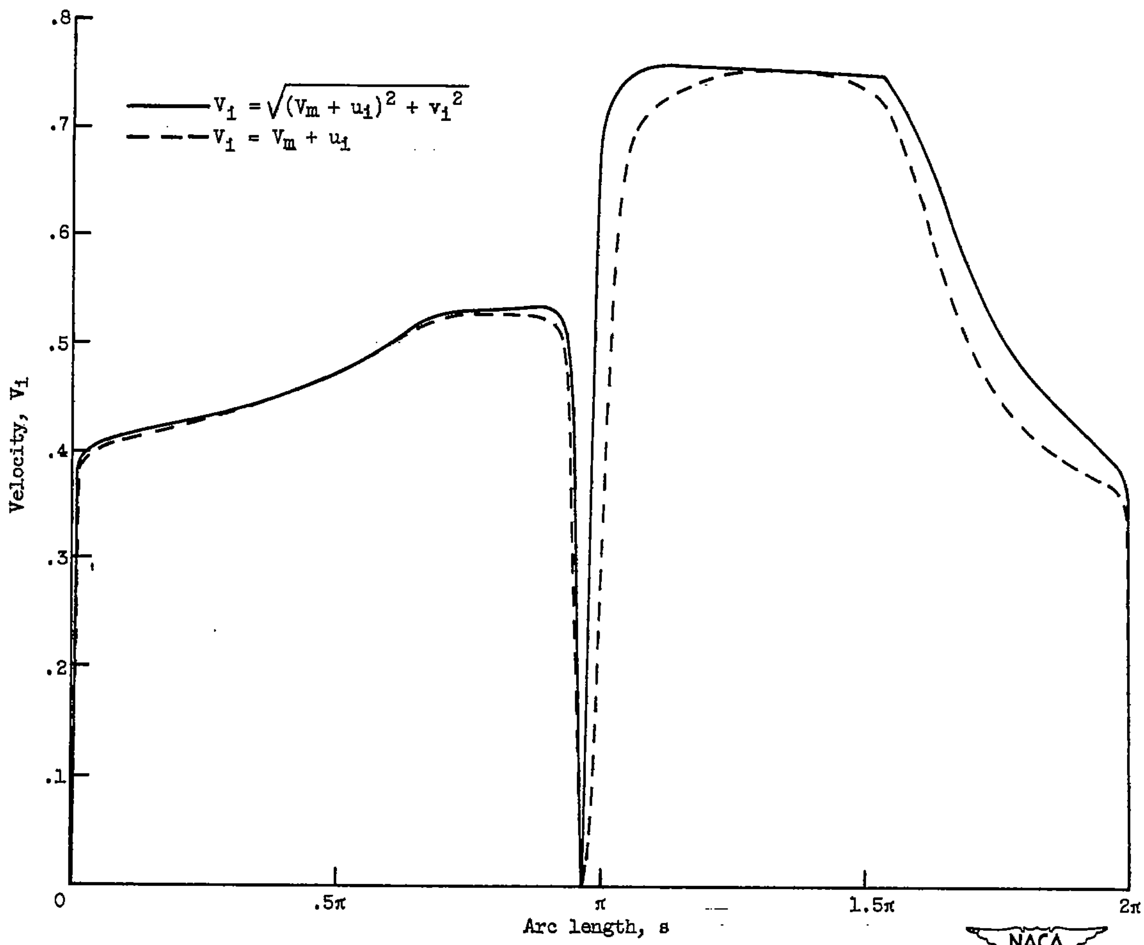

Figure 6. - Velocity distributions for exact incompressible flow and for linearized incompressible flow, showing error resulting from linearization.

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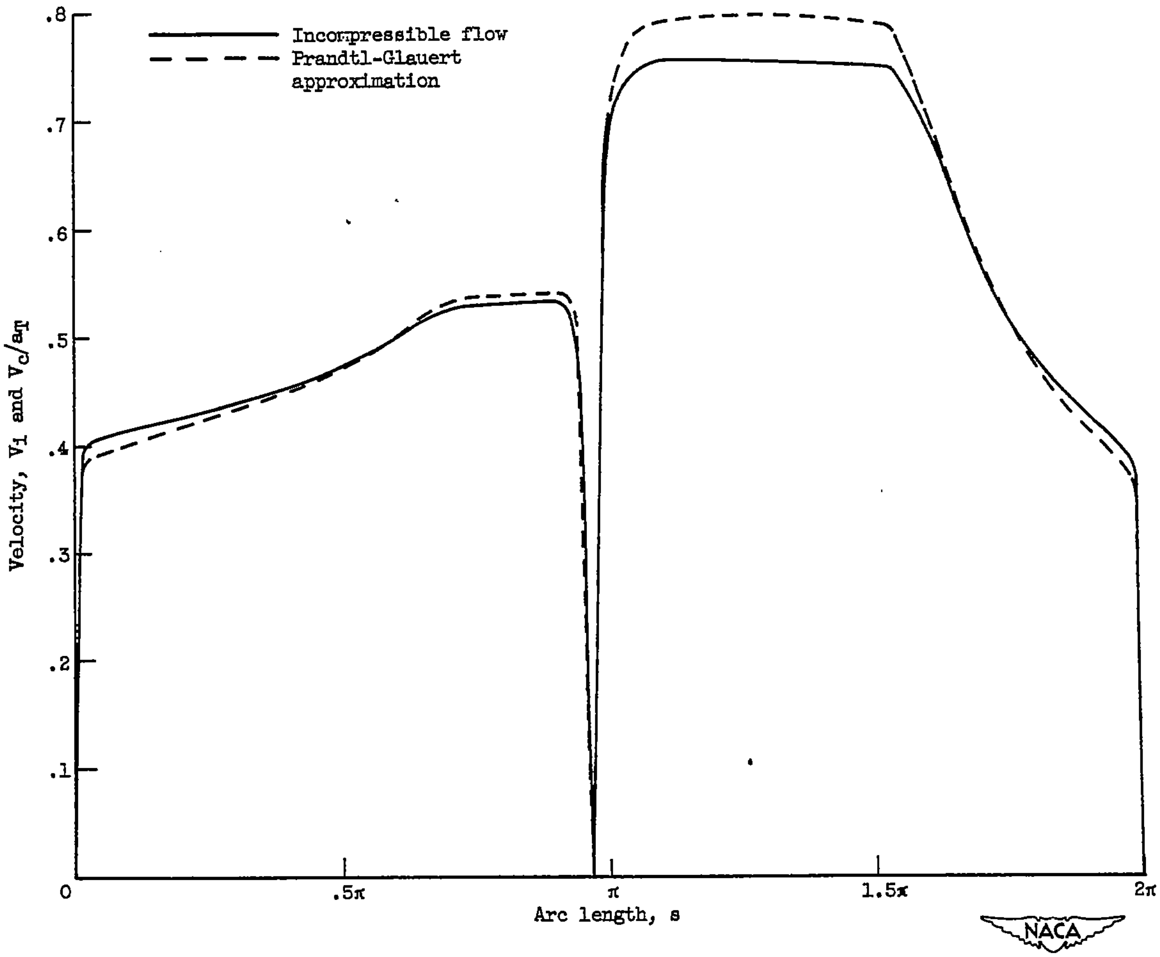

Figure 7. - Velocity distribution for incompressible flow and for Prandtl-Glauert approximation, showing magnitude of compressibility correction. Velocity $V = \sqrt{(V_m + u)^2 + v^2}$.

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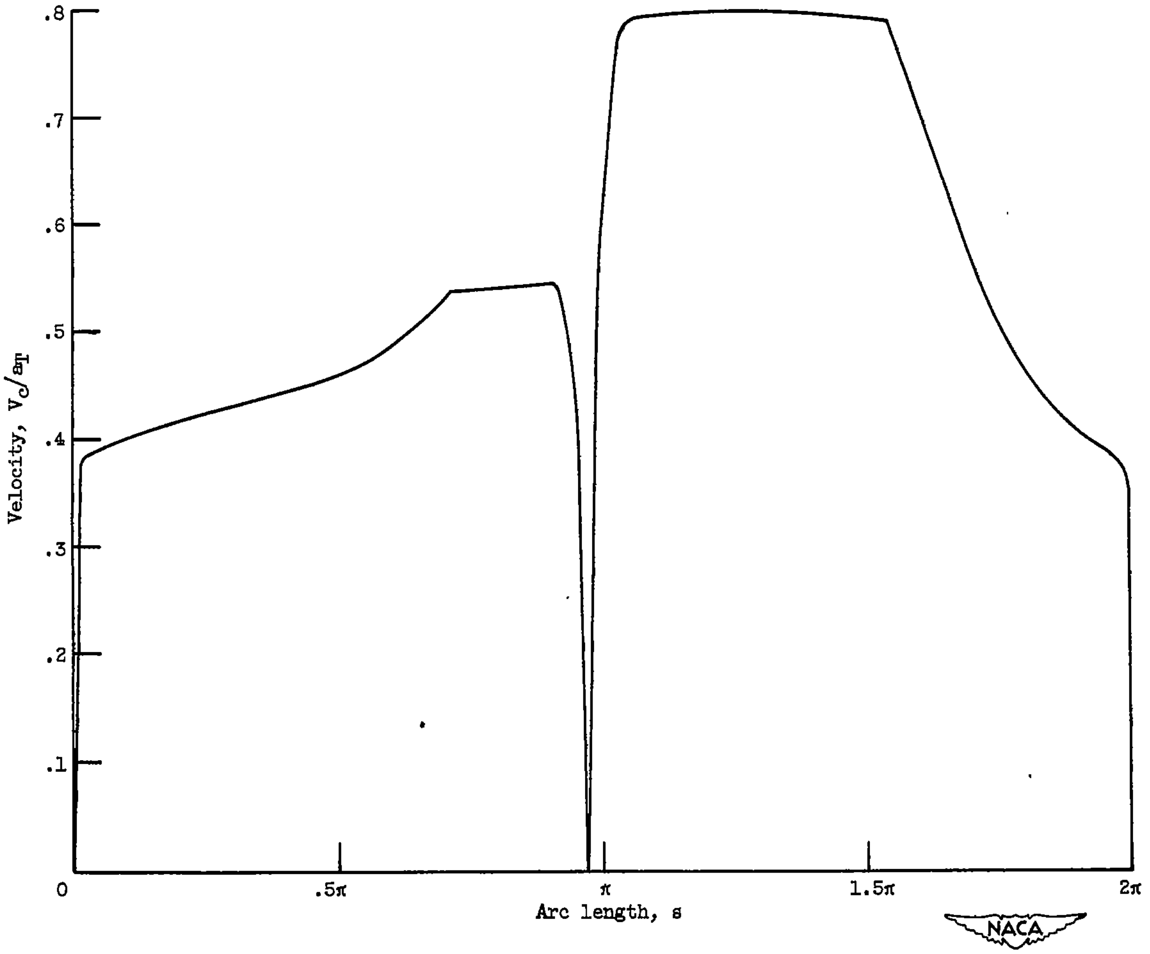

Figure 8. - Velocity distribution for approximation with dimensionless velocity proportional to that of linearized-pressure-volume flow.

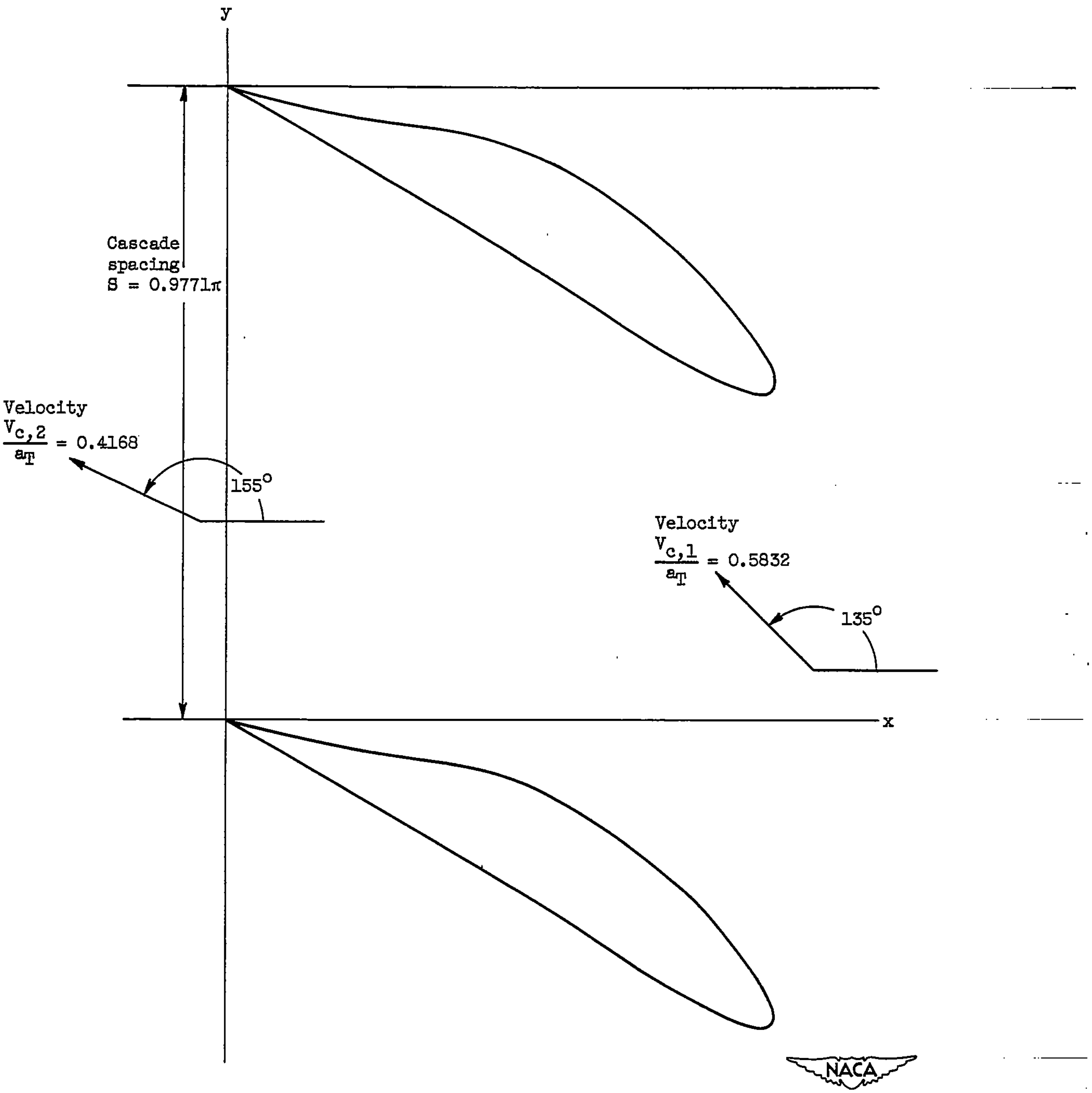

Figure 9. - Cascade geometry for approximation with dimensionless velocity proportional to that of linearized-pressure-volume flow. Mean Mach number, 0.5050; solidity, 0.9856.